Quadratic Action and Related Topics (SEMINAR ON PERMUTATION GROUPS AND RELATED TOPICS)

HO, CHAT-TIN

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CHAT-TIN HO
Univ. de Brasília

(visiting The University of Tsukuba)

Let M be a vector space over a field F and let $G \leq GL(M)$. An element σ of G acts quadratically on M provided the minimal polynomial of σ is $(X-1)^2$, i.e., $M(\sigma-1)^2 = 0$.

Example. $G = SL(M)$ and σ = transvection.

Question : What can we say about G if $G = \langle\sigma | \sigma$ acts quadratically on M .

To answer this question in general is quite difficult. Let us restrict now to case that G is a finite group and M is a finite dimensional vector space over the field F_p of p elements. We note that if $p = 2$, then any element of order 2 has the minimal polynomial $(X-1)^2$. So let us look at the case p = odd prime.

Definition. Let M be a finite dimensional vector space over F_p and $G \leq GL(M)$. Let $Q = \{\sigma \in G | 1 \neq \sigma, \sigma$ acts quadratically on M$\}$. We say that (G,M) is a quadratic pair for p if

(1) M is an irreducible faithful G-module.

(2) $G = \langle\sigma | \sigma \in Q\rangle$.

Let $d = \min_{\sigma \in Q} \{\dim M(\sigma-1)\}$ and $Q_d = \{\tau \in Q_d \mid \dim M(\tau-1) = d\}$. For $\sigma \in Q_d$, let $E(\sigma) = \{\tau \in Q \mid M(\sigma-1) = M(\tau-1), C_M(\sigma) = C_M(\tau)\} \cup \{1\}$. Then $E(\sigma)$ is an elementary abelian p-group. Let $\Sigma = \{E(\sigma) \mid \sigma \in Q_d\}$. We say that (G,M) is a quadratic pair for 3 whose root group has order 3 if $|E| = 3$ for all $E \in \Sigma$.

Lemma. Suppose (G,M) is a quadratic pair for 3 whose root group has order 3. Let $\sigma, \tau \in Q_d$. Then $<\sigma, \tau>$ is isomorphic to one of the following groups : SL(2,3) $SL(2,3) \times Z_3$, SL(2,5), Z_3, $Z_3 \times Z_3$, 3^{1+2} the group of order 27 exponent 3 and nilpotent class 2.

Theorem 1. Let (G,M) be a quadratic pair for p, p odd, such that G is quasisimple. If (G,M) is a quadratic pair for 3 whose root group has order 3, then we also assume that for some $E \in \Sigma$, the set $\{H \mid H \in \Sigma \text{ and } <E,H> \cong SL(2,3)\times Z_3\}$ is empty. Then G/Z(G) is isomorphic to one of the following groups :

(1) Groups of Lie type of odd characterisitc : An(q)($n \geq 2$ except in the case q =3 where we have $n \geq 3$), ${}^2A_n(q)$ $(n \geq 2)$, $B_n(q)$ $(n \geq 3)$, $C_n(q)$ $(n \geq 2)$, $D_n(q)$ $(n \geq 3)$, ${}^2D_n(q)$ $(n \geq 3)$, ${}^3D_4(q)$, $G_2(q)$, $F_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$ where $q = p^b$ for some integer b.

(2) Groups of Lie type of even characteristic : $PGU_n(2)$, $S_p(6,2)$, $D_4(2)$, $G_2(4)$.

(3) Alternating groups of degree ≥ 5.

(4) Sporadic groups : HJ, Suz, CO_1.

Furthermore we have p = 3 whenever (2), (3) or (4) holds.

The proof of this theorem appears in Comm. in alg. 1975 (3), J. of alg. Vol. 41 and Vol. 43, 1976. A sketch of the proof may be found in Bull. A.M.S. 1976. Originally Thompson proved the theorem for $p \geq 5$ without the quasisimplicity.

For some geometrical reason we sould like to remove both the irreduciblity and quasisimplicity of the theorem. In this direction let me state the following result.

Theorem 2. (J. of alg. 197?) Let M be a vector space over the field F_p, when $p \geq 5$. Suppose every element of order p acts quadratically on M and $G = \langle\sigma | \sigma^p = 1\rangle$. Then $G = O_p(G) \circ S_1 \circ \quad \circ S_n$, where $S_i \cong SL(2,p^a)$ if $S_i = 1$. Furthermore $nil(O_p(G)) \leq 2$ and $\exp(O_p(G)) \leq p$.

If we assume in addition that $|M(\sigma-1)| = |M(\tau-1)|$ for any two nontrivial elements of order p in G., then either G is an elementary abelian p-group or $G \cong SL(2,p^a)$.

Remark. $SL(2,3) \times Z_3$ has a faithful module satisfies all assumptions of the last part of Theorem 2.

A translation plane of order p^d, p a prime, may be interpreted as follows : The points are the elements of a 2d-dimensional vector space M over F_p. The lines through the zero vector form a spread : a class of d-dimensional vector space of M such that each non-zero vector belongs to exactly one member of the class. The member of this class are called components of the spread (or 0-lines). The other lines are translates of the 0-lines. A collineation fixing 0 is a linear transformation leaves the spread invariaut. An affine elation is a collineation whose fixed-point is a component. A Baer p-element is a collineation of order p such that its fixed-point set is a square root subplane. D. Foulser (1974) shows that when p = odd a collineation fixing 0 is an affine elation or Baer p-element if and only if as a linear transformation of M its minimal polynomial is $(X-1)^2$. T. Ostrom (1970) and Ch. Hering (1972) have studied the group generated by affine elations fixing 0 and D. Foulser (1974) studied the group generated by Baer p-elements fixing 0. Their study leads to the following result.

Theorem. 3 Let p be an odd prime and M be a finite dimensional vector space over F_p such that dim M = 2d. Let $G \leq GL(M)$ suppose every element σ of order p

<u>acts quadratically on</u> M <u>and</u> $\dim M(\sigma-1) = d$. <u>If</u> $G = \langle\sigma \mid \sigma^p = 1\rangle$, <u>then one of the following holds</u> :

(a) G <u>is an elementary abelian</u> p-<u>group</u>.

(b) $G \cong SL(2,q)$, <u>where</u> $q = p^a$.

(c) $p = 3$ <u>and</u> $G = TO_2(G)$, <u>where</u> $|T| = 3$ <u>and</u> $O_2(G)$ <u>is special also</u> $[T, O_2(G)] = O_2(G)$ <u>and</u> $G \geq S \cong SL(2,3)$.

(d) $p = 3$ <u>and</u> $G = SO_2(G)$, <u>where</u> $S \cong SL(2,5)$. $\mathrm{Nilp}(O_2(G)) \leq 2$, $\exp(O_2(G)/Z(O_2(G))) \leq 2$, $\exp($<u>Frattini subgroup</u> of $O_2(G)) \leq 2$ <u>and</u> $\exp(O_2(G)) \leq 4$.

In the case $p = 2$, this study leads to the following definition. Let M be a vector space over F_2. We call an involution σ a free involution if $\dim C_M(\sigma) = \frac{1}{2}\dim M$.

<u>Question</u> : <u>Which abstract groups have a faithful</u> F_2-<u>module such that each involution is free</u> ?

<u>Answer</u> : <u>Groups of even order</u>.

We can see this by looking at the regular permutation F_2-module. However some results have been obtained. (See Hering and HO 'On free involutions in linear groups and collineation groups of translation planes', Preprint.)

Let X be a square nonsingular matrix and let

$$H = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix} \right\rangle .$$

What is H ? This is a generalized situation of Dickson's theorem in the finite case. It will be very interest to know what can be said in the case when charateristic is 0.

Let p be a prime and G a finite group which satisfies the following condition.

(TIp) : <u>two different Sylow</u> p-<u>subgroups</u> intersect trivially.

Suzuki (1964) treated the case $p = 2$. When p is an odd prime, it seems quite difficult to describe all possibilities. We say that G is a (Qp)-group if G satisfies the following:

(Qp) : *There exists a finite vector space* M *over* F_p, *such that* M *is a faithful* G-*module and some nontrivial element of* G *has minimal polynomial* $(X-1)^2$ *over* M.

Theorem 4. Let p *be an odd prime and let* G *be a finite group which satisfies conditions* (TIp) *and* (Qp). *Then one of the following holds:*

(a) *a Sylow* p-*group of* G *is normal*.

(b) $G \triangleright G_1 \triangleright G_2 \triangleright 1$, *where* $G_2 = Z(G_1)$. *Both* G/G_1, G_2 *are* p'-*groups*. $G_1/G_2 \cong L_2(p^n)$ *or* $U_3(p^n)$ *for some integer* n.

(c) $p = 3$, $G \triangleright G_1 \triangleright G_2 \triangleright 1$, *where* $G_2 = O_2(G_1)$, *and* G/G_1 *is a* $3'$-*group*. *Furthermore* $G_1/G_2 \cong Z_3$ *or* A_5.